Modified Perturbation Method for Spinor Electrodynamics*

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The zeroth approximation of a modified perturbation theory of photons and electrons is constructed by the application of Salam's gauge technique.

1. INTRODUCTION

THE application to spinor electrodynamics of the gauge technique developed by Salam[1] is the purpose of this article. While conventional perturbation methods are applicable to the spinor theory (which is not the case with vector electrodynamics) it is interesting to see that the new method also is suitable and that it may hold out the possibility of some improvements in practical calculations. The first objective of the new approach is the selection, through the judicious use of gauge invariance, of a suitable "zeroth approximation." It is this problem to which the present note is directed.

Basically, the approach consists in constructing strictly gauge invariant approximations to the basic Green's functions, the fermion and photon propagators and the vertex part. In fact one can start the approximations by requiring the propagators to satisfy 2-particle unitarity while the vertex part satisfies the Ward-Takahashi identity. There is, of course, considerable latitude in the choice of vertex part at this stage since the Ward-Takahashi identity determines fully only its longitudinal component. In the zeroth approximation —which is the only one considered here—one is guided by the requirements that the (proper) vertex part should converge to zero asymptotically and that it should be sufficiently simple to be of use in practical calculations.

2. THE ZEROTH APPROXIMATION

The fermion and photon propagators may be expressed by

$$S_F^{-1}(p)=(p-m)Z(p)\,, \tag{2.1}$$

and

$$D_{F\mu\nu}^{-1}(t)=(t^2-\mu^2)Z_3(t^2)d_{\mu\nu}(t) -(t^2-\lambda^2)(Z_3\mu_0^2/\lambda^2)e_{\mu\nu}(t)\,, \tag{2.2}$$

respectively, where $Z^{-1}(p)$ and $Z_3^{-1}(t^2)$ are given by the dispersion integrals

$$Z^{-1}(p)=1+(p-m)\left(\int_{-\infty}^{-(m+\mu)}+\int_{(m+\mu)}^{\infty}\right)d\kappa\frac{\rho(\kappa)}{p-\kappa}, \tag{2.3}$$

and

$$Z_3^{-1}(t^2)=1+(t^2-\mu^2)\int_{(2m)^2}^{\infty}dk^2\frac{\rho_3(k^2)}{t^2-k^2}, \tag{2.4}$$

with $\rho(\kappa)=\epsilon(\kappa)(\kappa\rho_1(\kappa^2)+m\rho_2(\kappa^2))$, $\rho_{1,2,3}$ being the Lehmann weights.[2] The expression for $D_{F\mu\nu}$ is that of Matthews and Feldman[3] describing a massive gauge particle: the transverse part, proportional to $d_{\mu\nu}=-g_{\mu\nu}+t_\mu t_\nu/t^2$, having mass μ^2, and the longitudinal part, proportional to $e_{\mu\nu}=t_\mu t_\nu/t^2$, having mass λ^2. At the end of the calculation one can take the limit λ^2, $\mu^2\to 0$ with $\lambda^2/\mu^2\to a$, a constant characteristic of the gauge.

The Ward-Takahashi identity takes the form

$$(1/e)(p-p')\mu\Gamma\mu(p,p')=S_F^{-1}(p)-S_F^{-1}(p')\,, \tag{2.5}$$

where Γ_μ denotes the proper vertex part and e the electric charge. The particular solution, Γ^A, of (2.5) which seems the most appropriate at this stage is obtained by noticing that

$$\frac{1}{p-\kappa}-\frac{1}{p'-\kappa}=-(p-p')_\mu\frac{1}{p-\kappa}\gamma_\mu\frac{1}{p'-\kappa}$$

whence, from (2.1) and (2.3)

$$\begin{aligned}S_F^{-1}(p)-S_F^{-1}(p')&=-S_F^{-1}(p)(S_F(p)-S_F(p'))S_F^{-1}(p')\\ &=(p-p')_\mu S_F^{-1}(p)\left(\frac{1}{p-m}\gamma_\mu\frac{1}{p'-m}+\int d\kappa\rho(k)\frac{1}{p-\kappa}\gamma_\mu\frac{1}{p'-\kappa}\right)S_F^{-1}(p')\\ &=(p-p')_\mu\left[Z(p)\gamma_\mu Z(p')+S_F^{-1}(p)\times\int d\kappa\rho(\kappa)\frac{1}{p-\kappa}\gamma_\mu\frac{1}{p'-\kappa}S_F^{-1}(p')\right].\end{aligned} \tag{2.6}$$

Following the method of Ref. 1, one need only insert into (2.6) between $(p-p')_\mu$ and the expression in

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[1] A. Salam, Phys. Rev. 130, 1287 (1963); A. Salam and R. Delbourgo, Preceding paper, Phys. Rev. 135, B1398 (1964).

[2] See, for example, S. Schweber, *An Introduction to Relativistic Quantum Field Theory* (Row Peterson & Company, Evanston, Illinois, 1961), p. 676.

[3] P. T. Matthews and G. Feldman, Phys. Rev. 130, 1287 (1963).

brackets a factor $[e_{\mu\nu}(t)-Z_3(t^2)a_{\mu\nu}(t)]$, where $t=p-p'$. This does not change (2.6) but it will be found to improve the asymptotic behavior of Γ^A, which is obtained by stripping off the factor $(p-p')_\mu$, namely,

$$\Gamma_\mu{}^A(p,p')=e[e_{\mu\nu}(t)-Z_3(t^2)d_{\mu\nu}(t)]\bigg[Z(p)\gamma_\nu Z(p') +S_F{}^{-1}(p)\int d\kappa\rho(\kappa)\frac{1}{p-\kappa}\gamma_\nu\frac{1}{p-\kappa}S_F{}^{-1}(p')\bigg]. \quad (2.7)$$

which is easily seen to be a solution of (2.5).

It remains now to determine $Z(p)$ and $Z_3(t^2)$ by imposing 2-particle unitarity on the propagators. The unitarity relations are given by

$$\rho(p)=\frac{1}{(2\pi)^3}\int dp'dt\delta(t+p'-p)\theta(p')\delta(p'^2-m^2) \times\theta(t)S_F(p)\Gamma_\mu(p,p')(p'+m)[\delta(t^2-\mu^2)d_{\mu\nu} -\delta(t^2-\lambda^2)(\lambda^2/\mu^2)e_{\mu\nu}]\bar{\Gamma}_\nu(p,p')\bar{S}(p), \quad (2.8)$$

and

$$\rho(t^2)=\frac{1}{(2\pi)^3}\int dpdp'\delta(p+p'-t)\theta(p)\delta(p^2-m^2)\theta(p') \times\delta(p^2-m^2)[1/(t^2-\mu^2)Z_3(t^2)] \times\tfrac{1}{3}\,\mathrm{tr}[(p+m)\Gamma_\mu(p,-p')(-p'+m)\bar{\Gamma}_\mu(p,-p')] \times[1/(t^2-\mu^2)Z_3{}^*(t^2)], \quad (2.9)$$

where $\rho(p)=p\rho_1+m\rho_2$. If for Γ_μ in (2.7), (2.8) one substitutes $\Gamma_\mu{}^A$ given by (2.6), then, since

$$\Gamma_\mu{}^A(p'=m,\, t^2=\mu^2)=eZ(p)\gamma_\mu \quad\text{and}\quad \Gamma_\mu{}^A(p=m,\, p'=m)=eZ_3(t^2)\gamma_\mu \quad (2.10)$$

the structure disappears from the expressions for $\rho_{1,2,3}$ which reduce to those given by second-order perturbation theory, namely,

$$\rho_1(p^2)=\alpha\frac{(p^2+m^2)^2a-2m^2p^2(a+3)}{2(p^2)^2(p^2-m^2)}$$
$$\rho_2(p^2)=\alpha\frac{(a-3)(p^2+m^2)}{2p^2(p^2-m^2)} \quad (2.11)$$
$$\rho_3(t^2)=\alpha\frac{2}{3}\frac{t^2+2m^2}{(t^2)^2}\left(\frac{t^2-4m^2}{t^2}\right)^{1/2}$$

where $2\pi\alpha$ denotes the fine structure constant $e^2/4\pi$. Inserting these results into the dispersion integrals (2.3) (2.4) one finds in the symptotic region:

$$\frac{1}{Z(p)}\sim1+\frac{\alpha}{2p^2}(p-m)(pa\ln p^2+m(a-3)\ln p^2)$$
$$\frac{1}{Z_3(t^2)}\sim1+\tfrac{2}{3}\alpha\ln t^2. \quad (2.12)$$

Comparing these expressions with the definitions of the renormalization constants,

$$\lim(p-m)Z(p)\sim(p-m_0)Z_2,$$
$$\lim Z_3(t^2)\sim Z_3, \quad (2.13)$$

one sees that the approximation yields

$$Z_1=Z_2=Z_3=0 \quad\text{and}\quad m_0=[(a-3)/a]m. \quad (2.14)$$

While the results obtained for $Z^{-1}(p)$ and $Z_3{}^{-1}(t^2)$ coincide in this approximation with the usual perturbation theory expressions, it is clear that Γ^A and S_F now form the basis for subsequent approximations and these will be the subject of a further paper. Provided one does not discard the higher powers of α in Γ^A it will converge to zero for large momenta.

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